Journal of Sound and Vibration (1999) **223**(2), 255–271
Article No. jsvi.1998.2124, available online at http://www.idealibrary.com on IDEAL®

APPROXIMATE ANALYTICAL EXPRESSIONS FOR THE STOCHASTIC RESPONSE OF A BILINEAR HYSTERETIC OSCILLATOR WITH LOW YIELD LEVELS

R. C. MICALETTI AND A. S. ÇAKMAK
Department of Civil Engineering and Operations Research, Princeton University, Princeton, NJ 08540, U.S.A.

S. R. K. NIELSEN
Department of Building Technology and Structural Engineering, Aalborg University, DK-9000 Aalborg, Denmark

AND

H. U. KÖYLÜOĞLU
Oliver, Wyman & Co., 666 5th Avenue, 16th Floor, New York, 10103, U.S.A.

(*Received* 11 *June* 1997, *and in final form* 30 *November* 1998)

Differential equations are derived which exactly govern the evolution of the second order response moments of a single-degree-of-freedom (SDOF) bilinear hysteretic oscillator subject to stationary Gaussian white noise excitation. Then, considering cases for which response stationarity will be achieved, i.e., excluding the case of an elastic–perfectly-plastic oscillator, algebraic equations for the response moments are found. By the nature of the problem, these moments depend on the probability of the oscillator being in the plastic state. Upon considering oscillators with low yield levels and using analytically available information, physical reasoning, and approximations supported by empirical observation, an equation for the probability of the oscillator being in the plastic state is derived. Upon numerical solution of this equation, analytical approximations to the response moments can be obtained. All analytical, approximate, and numerical results are verified by extensive Monte Carlo simulations.

1. INTRODUCTION

Over the last few decades, the problems of predicting the response and reliability of hysteretic systems subject to random excitation have received considerable attention. While this attention has mainly stemmed from the engineering usefulness of hysteretic systems in the modelling of actual physical and mechanical phenomena, it has also been due to the inherent difficulty in obtaining exact closed-form solutions to these problems. As a consequence,

0022–460X/99/220255 + 17 $30.00/0

various approximate analytical procedures for handling the non-linearity and non-analyticity of hysteretic systems have been proposed. Among the approximate analytical methods most commonly used include Markov methods [1–7, 31], equivalent linearization [8–17], equivalent non-linearization with cumulant-neglect closure [18], the associated linear oscillator approach [19–26], and the Slepian process approach [27–29]. (It should be noted that due to the enormous amounts of literature on hysteretic systems, the foregoing is by no means an exhaustive list of references on the subject.) While most of the above-cited techniques produce good to excellent results for weakly to moderately non-linear oscillators, when highly non-linear oscillators are considered, such as when the yield level is low relative to the standard deviation of the corresponding linear oscillator and/or the secondary to primary stiffness is small, the accuracy of most of the aforementioned techniques breaks down or the methods are not justifiably applicable (i.e., the assumptions underlying the methods are not valid). In such cases, Monte Carlo methods are often the only recourse to solving the problem.

In this paper, a bilinear hysteretic oscillator is considered subject to stationary Gaussian white noise excitation. Such an oscillator is often used as an idealized model of a simple structure undergoing earthquake excitations. As such, it may often be necessary to determine approximate, yet accurate, statistics of the oscillator's stationary response—especially the standard deviation of displacement—quickly and efficiently. To this end, a system of differential equations is derived that exactly governs the second order response moments of the bilinear hysteretic oscillator. Then, considering cases of response stationarity, that is, excluding the case of an elasto–plastic oscillator, the differential equations can be reduced to algebraic equations that depend not only on the unconditional moments of response but also on the probability of the oscillator being in the plastic state as well as on response moments conditioned on the oscillator being in the plastic state. Using physical reasoning supported by empirical observation for the case of low yield levels, *a priori* bounds on the conditional response moments are obtained. These bounds in conjunction with analytical and empirical approximations of unprovided response moments lead to an equation for the probability of being in the plastic state. Upon numerical solution of this equation, analytical approximations of the unconditional standard deviations of velocity and displacement follow immediately. The accuracy of these approximations as well as the accuracy of all analytical, approximate, and numerical results contained herein is verified through extensive Monte Carlo simulations. In addition, a comparison is made between results from the proposed method and those found by the method of stochastic averaging.

2. MOMENTS EQUATIONS

Consider the following equations of motion for a SDOF bilinear hysteretic oscillator excited by Gaussian white noise [30],

$$\ddot{X} + 2\xi\omega_0\dot{X} + \omega_0^2(\alpha X + (1-\alpha)Z) = W(t), \tag{1}$$

$$\dot{Z} = \dot{X}\{1 - H(\dot{X})H(Z - u) - H(-\dot{X})H(-Z - u)\}, \tag{2}$$

$$X(0) = \dot{X}(0) = Z(0) = 0, \tag{3}$$

where X, $\dot{X}$, and $\ddot{X}$ denote, respectively, the random displacement, velocity, and acceleration of the oscillator; Z, the random hysteretic component of the response; $W(t)$, a zero-mean Gaussian white noise excitation with double-sided spectral density S_0; ξ, the damping ratio; ω_0, the circular eigenfrequency of the corresponding linear oscillator; α, the secondary to primary stiffness ratio; u, the positive yield level of the oscillator; and $H(\cdot)$, a Heaviside step-function defined as

$$H(y) = \left\{ \begin{array}{ll} 1, & \text{if } y \geqslant 0; \\ 0, & \text{otherwise.} \end{array} \right\} \tag{4}$$

In state vector form, the foregoing equations can be written as

$$\mathrm{d}\mathbf{Y} = \mathbf{a}(\mathbf{Y})\,\mathrm{d}t + \mathbf{b}\,\mathrm{d}B(t), \tag{5}$$

where

$$\mathbf{Y} = \begin{bmatrix} X \\ \dot{X} \\ Z \end{bmatrix}, \quad \mathbf{a}(\mathbf{Y}) = \begin{bmatrix} \dot{X} \\ -2\xi\omega_0\dot{X} - \omega_0^2(\alpha X + (1-\alpha)Z) \\ \dot{X}\{1 - H(\dot{X})H(Z-u) - H(-\dot{X})H(-Z-u)\} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

and $B(t)$ is a Wiener process. The vector $\mathbf{a}$ is referred to as the drift vector and the vector $\mathbf{b}$, in general, is known as the diffusion matrix (here it reduces to a vector).

Equation (5) represents a vector Markov diffusion process interpreted in the Itô sense and as such, Itô's differential formula can be applied to derive equations that govern the evolution of the response moments. Itô's differential formula for an arbitrary, well-behaved function $f(\mathbf{Y}, t)$ of the state vector $\mathbf{Y}$ and time t is given by

$$\mathrm{d}f = \frac{\partial f}{\partial t}\,\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial f}{\partial Y_i}\,\mathrm{d}Y_i + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial Y_i \partial Y_j}\,\mathrm{d}Y_i\,\mathrm{d}Y_j. \tag{7}$$

By substituting equation (5) into equation (7), letting $f(\mathbf{Y}, t) = Y_{i_1}^{m_1} Y_{i_2}^{m_2} \cdots Y_{i_k}^{m_k}$, and taking the expectation, differential equations for the mth order joint moments of response quantities can be derived, where $m = \Sigma_j m_j$.

The resulting general expressions for first and second order joint moments are given by

$$\dot{\mu}_i = E[a_i], \quad \dot{\kappa}_{ij} = E[a_i Y_j] + E[a_j Y_i] + b_i b_j, \tag{8, 9}$$

where $\mathrm{E}[\cdot]$ is the expectation operator, $\mu_i = \mathrm{E}[Y_i]$ and $\kappa_{ij} = \mathrm{E}[Y_i Y_j]$, and Y_j, a_j, and b_j denote the jth components of the vectors $\mathbf{Y}$, $\mathbf{a}$, and $\mathbf{b}$, respectively. Since consideration of moments higher than order two would lead to significantly more equations and, consequently, analytic intractability, first and second order response moments will be considered herein. (For a given problem, it will be left up to the analyst to decide whether these moments provide useful information.)

Due to the asymmetry of the drift vector, namely,

$$\mathbf{a}(\mathbf{Y}) = -\mathbf{a}(-\mathbf{Y}), \tag{10}$$

$\mathrm{E}[a_i] = 0$ for each i as a result of the zero initial conditions. Then, equation (8) implies that the vector of first order response quantities is identically zero, i.e., $\mu_i = 0$ for each i. Hence, only second order moments will be considered in what follows.

Substituting equation (6) into equation (9) yields the following set of differential equations governing the time evolution of the second order joint moments of response (dependence of the joint moments on time is suppressed for notational convenience):

$$\begin{aligned}
&\dot{\kappa}_{11} = 2\kappa_{12}, \quad \dot{\kappa}_{12} = \kappa_{22} - 2\xi\omega_0\kappa_{12} - \omega_0^2(\alpha\kappa_{11} + (1-\alpha)\kappa_{13}),\\
&\dot{\kappa}_{13} = \kappa_{23} + \kappa_{12} - \mathrm{E}[X\dot{X}H(\dot{X})\ H\ (Z-u)] - \mathrm{E}[X\dot{X}H(-\dot{X})H(-Z-u)],\\
&\dot{\kappa}_{22} = -4\xi\omega_0\kappa_{22} - 2\omega_0^2(\alpha\kappa_{12} + (1-\alpha)\kappa_{23}) + 2\pi S_0,\\
&\dot{\kappa}_{23} = -2\xi\omega_0\kappa_{23} - \omega_0^2(\alpha\kappa_{13} + (1-\alpha)\kappa_{33}) + \kappa_{22}\\
&\qquad - E[\dot{X}^2 H(\dot{X})H(Z-u)] - E[\dot{X}^2 H(-\dot{X})H(-Z-u)],\\
&\dot{\kappa}_{33} = 2\kappa_{23} - 2E[\dot{X}ZH(\dot{X})H(Z-u)] - 2E[\dot{X}ZH(-\dot{X})H(-Z-u)],
\end{aligned} \tag{11}$$

where

$$\kappa_{11} = E[X^2], \quad \kappa_{12} = E[X\dot{X}], \quad \kappa_{13} = E[XZ], \quad \kappa_{22} = E[\dot{X}^2], \quad \kappa_{23} = E[\dot{X}Z]$$

and

$$\kappa_{33} = E[Z^2].$$

In most approximate analyses involving moment equations, the expectations above involving Heaviside functions are replaced by equivalent polynomial expansions of various orders. Here these expectations will be expressed as response moments conditioned on the oscillator being in the plastic state. To this end, let $p_{X\dot{X}Z}(x, \dot{x}, z)$ represent the joint probability density function of X, $\dot{X}$ and Z. This joint probability density is of the mixed type in the sense that X and $\dot{X}$ are continuous variables of infinite range whereas Z ranges only between $-u$ and u and has a finite probability of assuming the values $-u$ or u. In addition, let $P\{Z = u\}$ represent the probability that the oscillator has reached the positive yield state. Due to the symmetry of the oscillator, the probability of being in the positive yield state is equal to the probability of being in the negative yield state, i.e., $P\{Z = u\} = P\{Z = -u\}$. With these conventions established, consider the term $\mathrm{E}[X\dot{X}H(\dot{X})H(Z-u)]$. This term can be rewritten as

$$\begin{aligned} E[X\dot{X}H(\dot{X})H(Z-u)] &= \int_{-\infty}^{\infty} \mathrm{d}x\, x \int_{-\infty}^{\infty} \mathrm{d}\dot{x}\, \dot{x}\, H(\dot{x}) \int_{-u^-}^{u^+} \mathrm{d}z\, H(z-u) p_{X\dot{X}Z}(x, \dot{x}, z) \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, x \int_{0}^{\infty} \mathrm{d}\dot{x}\, \dot{x}\, p_{X\dot{X}|Z}(x, \dot{x}|z=u) P\{Z=u\} \\ &= P\{Z=u\} E[X\dot{X}|Z=u]. \end{aligned} \tag{12}$$

Intuitively, the result is clear, the term $X\dot{X}H(\dot{X})H(Z-u)$ is equal to $X\dot{X}$ if the system is in the positive yield state and zero otherwise. Note that the velocity of the oscillator is non-negative when $Z=u$. Due to the symmetry of the oscillator, expressions similar to equation (12) are obtained for the following expectations:

$$\begin{aligned} \mathrm{E}[X\dot{X}H(-\dot{X})H(-Z-u)] &= P\{Z=-u\}\mathrm{E}[X\dot{X}|Z=-u] = P\{Z=u\}\mathrm{E}[X\dot{X}|Z=u], \\ \mathrm{E}[\dot{X}^2 H(\dot{X})H(Z-u)] &= \mathrm{E}[\dot{X}^2 H(-\dot{X})H(-Z-u)] = P\{Z=u\}\mathrm{E}[\dot{X}^2|Z=u], \\ \mathrm{E}[\dot{X}ZH(\dot{X})H(Z-u)] &= \mathrm{E}[\dot{X}ZH(-\dot{X})H(-Z-u)] = uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]. \end{aligned} \tag{13}$$

Substituting the results of equations (12) and (13) into equation (11) and setting the left sides equal to zero, i.e., considering the state of stationarity, yields the following algebraic equations for the determination of the stationary response moments:

$$\begin{aligned} &0 = 2\kappa_{12}, \quad 0 = \kappa_{22} - 2\xi\omega_0\kappa_{12} - \omega_0^2(\alpha\kappa_{11} + (1-\alpha)\kappa_{13}), \\ &0 = \kappa_{23} + \kappa_{12} - 2P\{Z=u\}\mathrm{E}[X\dot{X}|Z=u], \\ &0 = -4\xi\omega_0\kappa_{22} - 2\omega_0^2(\alpha\kappa_{12} + (1-\alpha)\kappa_{23}) + 2\pi S_0, \\ &0 = -2\xi\omega_0\kappa_{23} - \omega_0^2(\alpha\kappa_{13} + (1-\alpha)\kappa_{33}) + \kappa_{22} - 2P\{Z=u\}\mathrm{E}[\dot{X}^2]|Z=u], \\ &0 = 2\kappa_{23} - 4uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]. \end{aligned} \tag{14}$$

The first of equations (14) eliminates κ_{12} from consideration. Further, both the third and last of equations (14) yield expressions for κ_{23}. Use will be made of the latter, as it requires only the knowledge of the marginal probability density of $\dot{X}$ when the system is in the plastic state, whereas the former requires information about the joint probability density of X and $\dot{X}$ in the plastic state. Note that by discarding the third of equation (14), four non-trivial/non-redundant equations for the eight unknowns κ_{11}, κ_{13}, κ_{22}, κ_{23}, κ_{33}, $\mathrm{E}[\dot{X}]Z=u]$, $\mathrm{E}[\dot{X}^2|Z=u]$, and $P\{Z=u\}$ remain. Hence extra information regarding the physical system, including the probability density of $\dot{X}$ during plastic excursions, is needed to obtain numerical results for the response moments.

3. MARGINAL PROBABILITY DENSITY FUNCTION OF VELOCITY DURING PLASTIC EXCURSIONS

Consider the stationary response of the corresponding linear SDOF dynamical system. Let u be an arbitrary positive distance from the origin. It is well-known from crossing theory, that if values of $\dot{X}$ are sampled at u-upcrossings, the empirical distribution of $\dot{X}$ tends to a Rayleigh distribution. That is, the probability density of $\dot{X}$ tends to one that has no probability mass arbitrarily near zero as the velocity of the system must be greater than zero in order to have a u-upcrossing.

From the equations of motion of the bilinear hysteretic oscillator, equation (1), it is seen that between yield-level excursions, the system acts as a linear oscillator. Thus, if the yield level is sufficiently high in relation to the standard deviation of the corresponding linear oscillator, it is reasonable to assume that between plastic excursions the oscillator's response renormalizes to Gaussianity (this idea forms the basis of the Slepian process approach) such that the distribution of $\dot{X}$ at yield-level upcrossings is approximately Rayleigh-distributed with mean $\sqrt{\pi/2}\sigma_{\dot{X}}$ and second moment $2\sigma^2_{\dot{X}}$, where $\sigma_{\dot{X}}$ is the unconditional standard deviation of velocity. However, this distribution does not account for the distribution of $\dot{X}$ for the system's entire sojourn to the plastic state. During plastic excursions, if white noise effects are neglected, the velocity of the oscillator monotonically decreases from its value at yield-level upcrossing to zero at which point the oscillator exits the plastic state. Physically, this means that the average value of $\dot{X}$ during plastic excursions is less than the average value of $\dot{X}$ at yield-level upcrossings. Further, as every yield-level upcrossing corresponds to a zero velocity yield-level downcrossing, it seems physically reasonable to expect the probability density of $\dot{X}$ during plastic excursions to have some probability mass arbitrarily near zero—possibly approaching a half-Gaussian probability density with mean $\sqrt{2/\pi}\sigma_{\dot{X}}$ and second moment $\sigma^2_{\dot{X}}$.

For low yield levels (in relation to the standard deviation of the corresponding linear oscillator), it is not the case that the system's response will renormalize to Gaussianity between plastic excursions. However, the velocity at a u-upcrossing still must be positive and this velocity will again steadily decrease to zero during plastic excursions such that the average velocity during plastic excursions is less than the average value at u-upcrossings. Thus, while the distribution of the velocity at u-upcrossings is not expected to be Rayleigh, it is conjectured that the mean and second moment of the Rayleigh distribution (with parameter $\sigma^2\dot{X}$ equal to the unconditional variance of velocity of the bilinear oscillator) form upper bounds on the mean and second moment of $\dot{X}$ during plastic excursions. Further, it is conjectured (again for low yield levels) that the mean and second moment of the half-Gaussian density (with parameter $\sigma^2_{\dot{X}}$ equal to the unconditional variance of velocity) form lower bounds on the mean and second moment of $\dot{X}$ during plastic excursions.

These conjectures are supported by results obtained through Monte Carlo simulations. A SDOF bilinear hysteretic with $\omega_0 = 1$, $\xi = 0{\cdot}05$, and $\alpha = 1/21$ was considered for various yield levels u. The intensity of the white noise excitation was prescribed so that the mean-square displacement of the corresponding linear

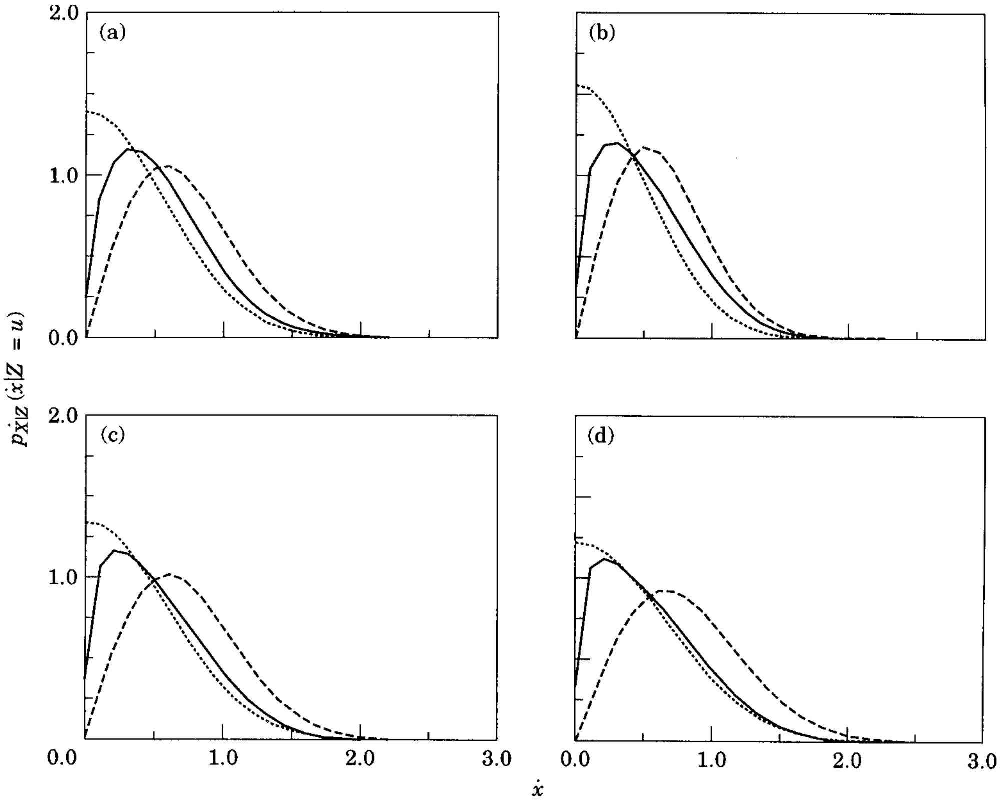

Figure 1. Conditional probability densities of $\dot{X}$ given $Z = u$; Monte Carlo simulations (—); half-Gaussian (·····); Rayleigh (- - -); (a) $u/\sigma_{X,0} = 0{\cdot}25$; (b) $u/\sigma_{X,0} = 0{\cdot}50$; (c) $u/\sigma_{X,0} = 0{\cdot}75$; (d) $u/\sigma_{X,0} = 1{\cdot}00$.

oscillator, $\sigma^2_{\dot{X},0}$, was 1, i.e., $S_0 = 2\xi\omega_0^3/\pi$. Thus, the yield level u is normalized in proportion to the standard deviation of the corresponding linear oscillator. Figure 1 shows the conditional probability density functions of $\dot{X}$ given that the system is in the positive yield state in comparison with Rayleigh and half-Gaussian densities for various values of $u/\sigma_{X,0}$. The means and second moments of the distributions are listed in Table 1. For ratios of $u/\sigma_{X,0}$ less than or equal to one, the means and second moments of the Rayleigh and half-Gaussian distributions form upper and lower bounds, respectively, on the means and

TABLE 1

Means and second moments of the probability densities appearing in Figure 1

	Means			Second moments		
$u/\sigma_{X,0}$	Half-Gaussian	Empirical	Rayleigh	Half-Gaussian	Empirical	Rayleigh
0·250	0·457	0·553	0·718	0·328	0·436	0·656
0·500	0·409	0·517	0·643	0·263	0·385	0·526
0·750	0·475	0·531	0·746	0·354	0·415	0·708
1·000	0·523	0·554	0·821	0·429	0·458	0·858

second moments of the empirical distributions of $\dot{X}$. Thus in the ensuing analysis, the terms $\mathrm{E}[\dot{X}|Z=u]$ and $\mathrm{E}[\dot{X}^2|Z=u]$ in equation (14) are taken to be bounded as follows:

$$\sqrt{2/\pi}\sigma_{\dot{X}} \leqslant \mathrm{E}[\dot{X}|Z=u] \leqslant \sqrt{\pi/2}\sigma_{\dot{X}}; \quad \sigma^2_{\dot{X}} \leqslant \mathrm{E}[\dot{X}^2|Z=u] \leqslant 2\sigma^2_{\dot{X}}. \tag{15, 16}$$

4. EQUATION FOR THE PROBABILITY OF BEING IN THE PLASTIC STATE

For ease of notation consider the following reformulation of equation (1):

$$\ddot{X} + a\dot{X} + bX + cZ = W(t), \tag{17}$$

where $a = 2\xi\omega_0$, $b = \omega_0^2\alpha$, and $c = \omega_0^2(1-\alpha)$. The non-trivial/non-redundant algebraic equations for the stationary response moments become

$$\begin{aligned} 0 &= \kappa_{22} - b\kappa_{11} - c\kappa_{13}, \quad 0 = \kappa_{22} + (c/a)\kappa_{23} - \pi S_0/a, \\ 0 &= -a\kappa_{23} - b\kappa_{13} - c\kappa_{33} + \kappa_{22} - 2P\{Z=u\}\mathrm{E}[\dot{X}^2|Z=u], \\ 0 &= 2\kappa_{23} - 4uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]. \end{aligned} \tag{18}$$

The last of equation (18) indicates that $\kappa_{23} = 2uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]$ which, given the bounds on $\mathrm{E}[\dot{X}|Z=u]$, can be written as

$$\kappa_{23} = 2uP\{Z=u\}d\sigma_{\dot{X}} = f\sigma_{\dot{X}}, \quad \sqrt{2/\pi} \leqslant d \leqslant \sqrt{\pi/2}, \tag{19}$$

where $f \equiv 2uP\{Z=u\}d$. Substituting this result into the second of equations (18) and using $\kappa_{22} = \mathrm{E}[\dot{X}^2] = \sigma^2_{\dot{X}}$ yields a quadratic equation in $\sigma_{\dot{X}}$, namely

$$\sigma^2_{\dot{X}} + (cf/a)\sigma_{\dot{X}} - \pi S_0/a = 0, \tag{20}$$

whose positive root is given by

$$\sigma_{\dot{X}} = -cf/2a + \sqrt{(cf/2a)^2 + \pi S_0/a}. \tag{21}$$

The standard deviation of velocity as given by equation (21) is a monotonically decreasing function of $udP\{Z=u\}$ for given system parameters. It follows then that the mean-square velocity, $\sigma^2_{\dot{X}}$, will also be a monotonically decreasing function of $udP\{Z=u\}$. When $u \to 0$, $P\{Z=u\} \to 0{\cdot}5$ and the system behaves as a linear oscillator with stiffness $\alpha\omega_0^2$. When $u \to \infty$, $P\{Z=u\} \to 0$ and the system behaves as a linear oscillator with stiffness ω_0^2. In both cases, $\sigma^2_{\dot{X}} = \pi S_0/2\xi\omega_0$ which forms an upper bound on the mean-square velocity. When $udP\{Z=u\}$ is non-zero, the oscillator's stiffness will be repeatedly softened upon its entries into the plastic state. This softening of the restoring force leads to oscillations of lower frequency and, consequently, lower velocity implying that the mean-square velocity will decrease.

Turning now to the third of equations (18) and using $\kappa_{22} = \sigma^2_{\dot{X}}$, $\kappa_{23} = f\sigma_{\dot{X}}$, and equation (16) results in

$$\sigma^2_{\dot{X}}(1-g) - af\sigma_{\dot{X}} = b\kappa_{13} + c\kappa_{33}, \tag{22}$$

where

$$g \equiv 2P\{Z=u\}e, \quad 1 \leqslant e \leqslant 2. \tag{23}$$

The significance of equation (22) is that the left side is entirely a function of known system constants, parameters that are bounded, and the probability of being in the plastic state. Consequently, if the right side of equation (22) *a priori* can be well-approximated in terms of system constants and/or the probability of being in the plastic state, the equation can be solved for $P\{Z=u\}$ with regard to the bounded constants d and e to give functioning (though not rigorous) bounds on $P\{Z=u\}$. Thus, the problem becomes one of determining valid approximations of κ_{13} and κ_{33}.

5. APPROXIMATIONS OF κ_{13} AND κ_{33}

First consider κ_{33}. Let $p_Z(z)$ be the marginal mixed probability density of z and let $\hat{p}_Z(z)$ represent the continuous part of $p_z(z)$ between $-u$ and u. Due to the symmetry of the oscillator, $p_Z(z)$ is symmetric about zero. Assuming $\hat{p}_Z(z)$ to be approximately uniformly distributed, i.e., $\hat{p}_Z(z) \approx (1/2u)(1-2P\{Z=u\})$, it follows that

$$\kappa_{33} = E[Z^2] = 2\int_0^{u^+} \mathrm{d}z\, z^2 p_Z(z) = u^2 2P\{Z=u\} + 2\int_0^{u} \mathrm{d}z\, z^2 \hat{p}_Z(z)$$
$$\approx u^2 2P\{Z=u\} + u^2 \tfrac{1}{3}(1-2P\{Z=u\}). \tag{24}$$

Figure 2 shows estimated exact values of κ_{33} as computed via Monte Carlo simulations in comparison with values predicted by equation (24) using the estimated exact values of $P\{\mathrm{Z}=u\}$ as determined by Monte Carlo simulations. As is evident, equation (24) provides a very close approximation to the actual value of κ_{33}, especially at low values of $u/\sigma_{X,0}$.

To determine an approximation for κ_{13}, first note that $X = Z + \Delta$ where Δ is the net plastic displacement of the oscillator. Thus,

$$\kappa_{13} = \mathrm{E}[XZ] = \mathrm{E}[Z^2] + \mathrm{E}[Z\Delta] \approx \mathrm{E}[Z^2] = \kappa_{33}, \tag{25}$$

where use was made of the fact, uncovered by extensive Monte Carlo simulations, that Z and Δ are approximately uncorrelated. This correlation is the difference between κ_{13} and κ_{33} but it is quite small for values of $\alpha \leqslant 0{\cdot}25$ (which covers most cases of practical interest) and reasonably small for $\alpha \geqslant 0{\cdot}50$ as can be seen in Figure 3.

Thus, in the sequel the right side of equation (22) is approximated by

$$b\kappa_{13} + c\kappa_{33} \approx b\kappa_{33} + c\kappa_{33} \approx (b+c)u^2(2P\{Z=u\} + \tfrac{1}{3}(1-2P\{Z=u\})). \tag{26}$$

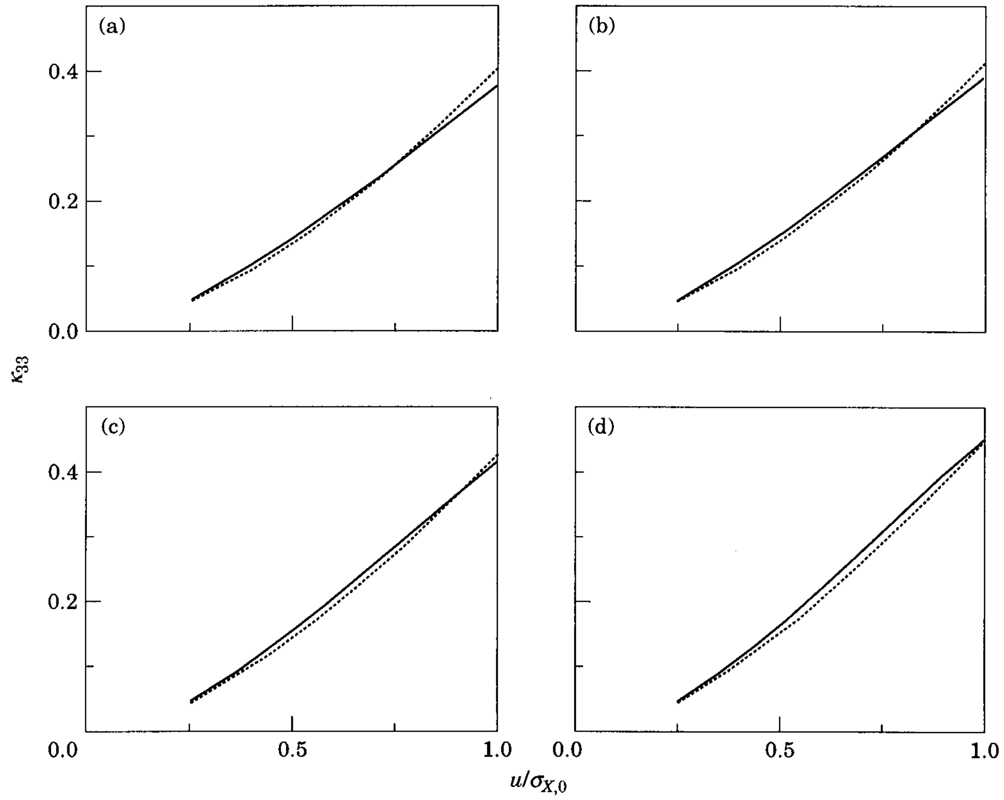

Figure 2. Comparison of approximate and estimated exact values of κ_{33}; Monte Carlo simulations (—); approximate values found using equation (25) (·····); (a) $\alpha = 0{\cdot}05$; (b) $\alpha = 0{\cdot}25$; (c) $\alpha = 0{\cdot}50$; (d) $\alpha = 0{\cdot}75$.

With the above results in conjunction with equation (22), the following equation for the probability of being in the plastic state is obtained:

$$\sigma_{\dot{X}}^2(1-g) - af\sigma_{\dot{X}} - (b+c)u^2(2P\{Z=u\} + \tfrac{1}{3}(1-2P\{Z=u\})) = 0, \qquad (27)$$

where $\sigma_{\dot{X}}$ is given in terms of $P\{Z=u\}$ by equation (21).

Equation (28) can be reduced to a quartic equation in $P\{Z=u\}$ and thus has an exact analytical solution. However, it is extremely lengthy as verified by the software program *Mathematica*. Thus, for purposes of numerical evaluation, equation (27) was solved numerically using *Mathematica*'s root-finding algorithm.

6. NUMERICAL EXAMPLES

To investigate the accuracy of the results given by equation (28) its dependence on the yield level, u, and the secondary to primary stiffness ratio, α, various parametric cases were considered. For each case, the intensity of the white noise, S_0, was prescribed so that the mean-square displacement of the corresponding linear oscillator was equal to one. Then u could be interpreted as the ratio of the yield level to the standard deviation of the corresponding linear

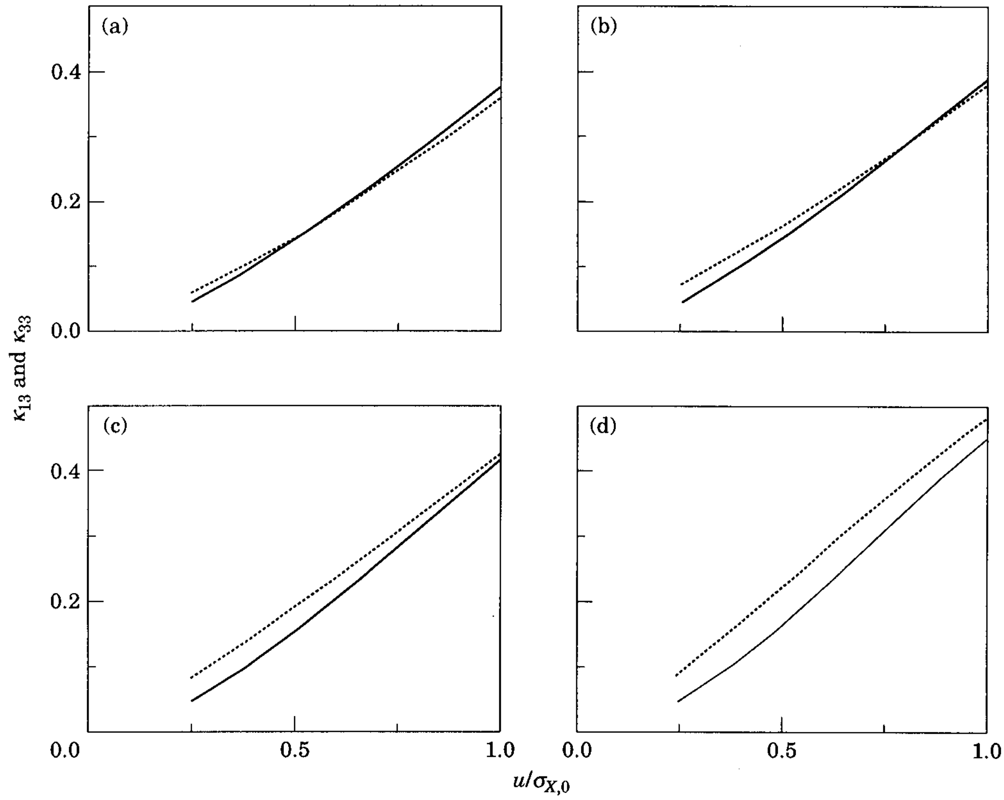

Figure 3. Comparison of estimated exact values of E[Z^2] (—) with estimated exact values of E[XZ] (·····) obtained by Monte Carlo simulations; (a) $\alpha = 0{\cdot}05$; (b) $\alpha = 0{\cdot}25$; (c) $\alpha = 0{\cdot}50$; (d) $\alpha = 0{\cdot}75$.

oscillator or, equivalently, the reciprocal of u could be viewed as the factor by which the white noise intensity was increased or decreased with respect to the linear case. Values of u considered ranged from 0·25 to 1·0 while values of α ranged from 0·05 to 0·75. For all cases considered, $\omega_0 = 1{\cdot}0\ \mathrm{s}^{-1}$ (without loss of generality as time can always be measured in units of $1/\omega_0$) and $\xi = 0{\cdot}05$.

In addition, recall that the parameters d and e enter equation (28) through f and g and these parameters are restricted to the intervals $\sqrt{2/\pi} \leqslant d \leqslant \sqrt{\pi/2}$, $1 \leqslant e \leqslant 2$. $P\{Z = u\}$ was evaluated in two different ways with respect to d and e, namely, by setting d and e to their extreme low values and extreme high values, respectively. For all cases of α considered, the values of $P\{Z = u\}$ found by setting d and e equal to their extreme low values ($\sqrt{2/\pi}$ and 1, respectively) were on the high side, that is, they formed upper bounds on the probability of being in the plastic state. These upper bounds became increasingly sharp as $u/\sigma_{X,0}$ approached 1. The opposite was true for the values of $P\{Z = u\}$ found by setting d and e equal to their extreme high values ($\sqrt{\pi/2}$ and 2, respectively). In this case, the results formed lower bounds for all values of $u/\sigma_{X,0}$. Similarly, these bounds became increasingly sharp as $u/\sigma_{X,0}$ approached 1. Further numerical analysis of equation (28) reveals that $P\{Z = u\}$ is inversely and monotonically related to both d and e. That is, for a fixed value of $u/\sigma_{X,0}$, when d and e take

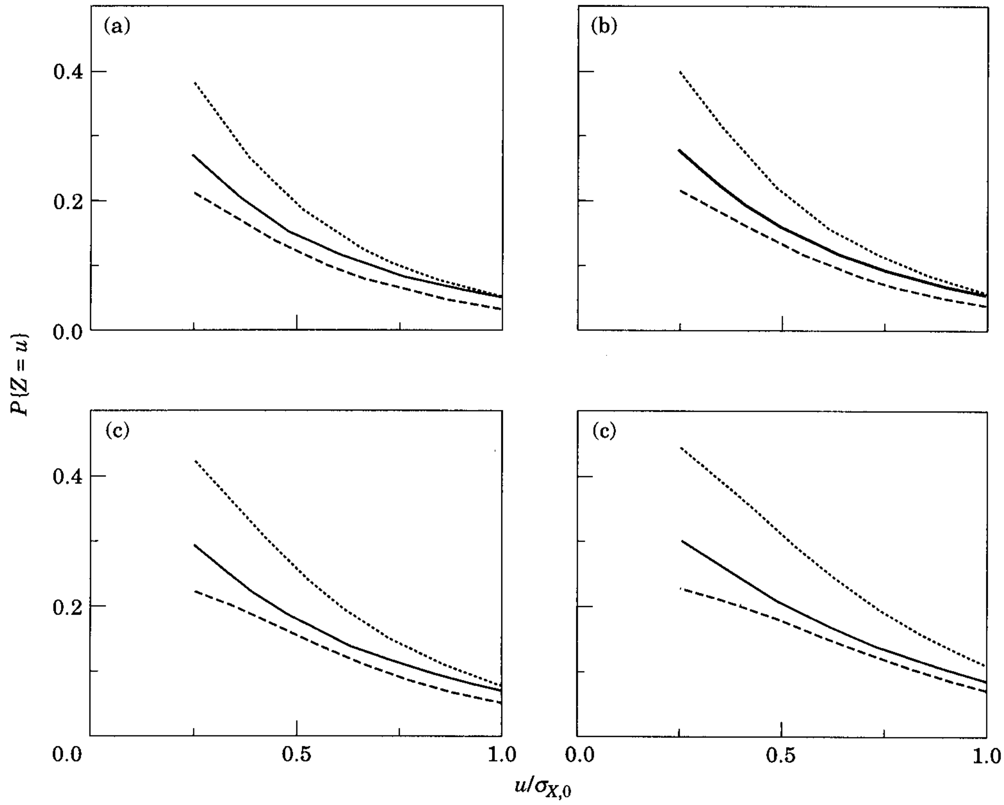

Figure 4. Comparison of estimated exact values of $P\{Z = u\}$ with solutions of equation (27); Monte Carlo simulations (—); upper bound (·····); lower bound (- - -); (a) $\alpha = 0{\cdot}05$; (b) $\alpha = 0{\cdot}25$; (c) $\alpha = 0{\cdot}50$; (d) $\alpha = 0{\cdot}75$.

their minimum values, $P\{Z = u\}$ attains its maximum value (and when d and e take their maximum values, $P\{Z = u\}$ attains its minimum value). Hence, no combination of values of d and e would produce higher or lower values than those found using the respective pairs of d and e given above.

Figure 4 shows comparisons of the estimated exact values of $P\{Z = u\}$ with results found by solving equation (28) using the extreme low and high values of the ordered pair of constants d and e, namely $(\sqrt{2/\pi}, 1)$ and $(\sqrt{\pi/2}, 2)$.

7. APPROXIMATION OF THE RESPONSE MOMENTS

With bounds on the probability of being in the plastic state established, the expression given by equation (21) can be used to determine an approximation of $\sigma_{\dot{X}}$. As before with the equation for $P\{Z = u\}$, the expression for $\sigma_{\dot{X}}$ can be evaluated using the extreme values of d and the corresponding values of $P\{Z = u\}$. These results are presented in Figure 5. The approximate results were essentially insensitive to the values of d used as long as the corresponding value of $P\{Z = u\}$ was used as well. This is because the variables d and $P\{Z = u\}$ appear as a product in equation (21) and recall that high values of d resulted in

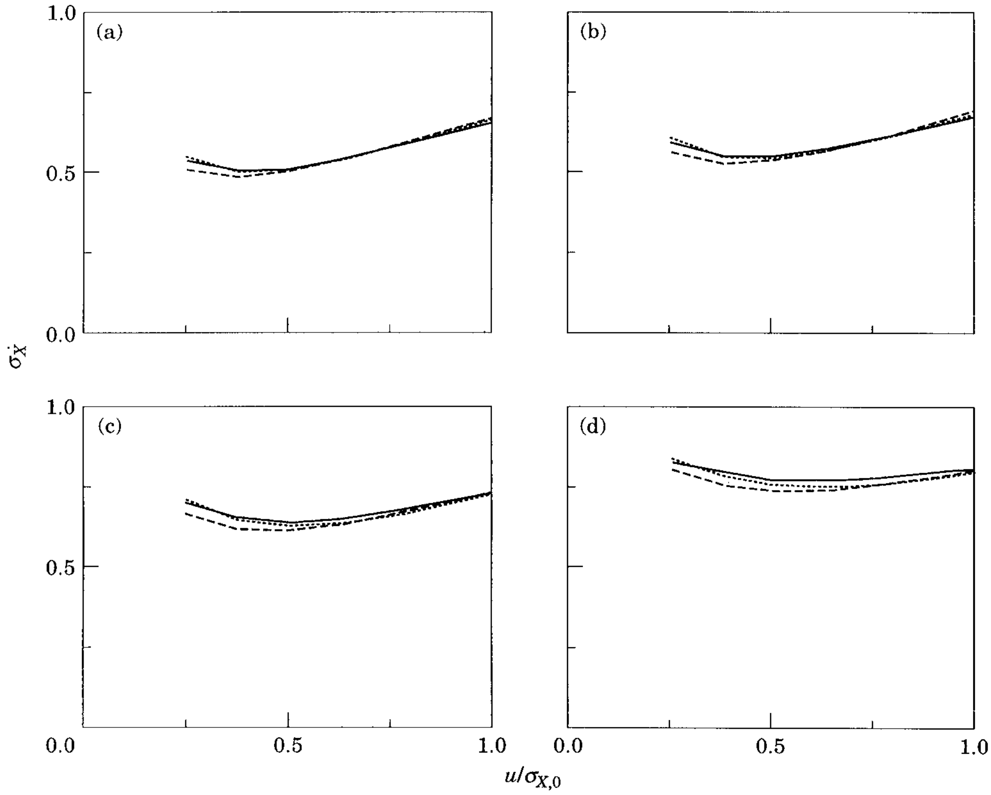

Figure 5. Comparison of estimated exact values of $\sigma_{\dot{X}}$ with approximations found using equation (21); Monte Carlo simulations (—); approximation 1 (·····); approximation 2 (- - -); (a) $\alpha = 0{\cdot}05$; (b) $\alpha = 0{\cdot}25$; (c) $\alpha = 0{\cdot}50$; (d) $\alpha = 0{\cdot}75$.

low values of $P\{Z = u\}$ and vice-versa. Hence, the products were approximately equal. In Figure 5, "Approximation 1" corresponds to using $d = \sqrt{\pi/2}$ (and the subsequent value of $P\{Z = u\}$) whereas "Approximation 2" corresponds to using $d = \sqrt{2/\pi}$.

Similarly, an approximation of σ_X can be computed using the first of equations (18) which can be written as

$$\sigma_X^2 = (1/b)(\sigma_{\dot{X}}^2 - c\kappa_{13}). \tag{28}$$

Making use of the approximation $\kappa_{13} \approx \kappa_{33} \approx u^2\ (2P\{Z = u\} + \frac{1}{3}(1 - 2P\{Z = u\}))$ yields

$$\sigma_X^2 = (1/b)\sigma_{\dot{X}}^2 - (c/b)u^2(2P\{Z = u\} + \tfrac{1}{3}(1 - 2P\{Z = u\})). \tag{29}$$

Once again, in light of the upper and lower bounds on $P\{Z = u\}$ there are multiple ways in which to evaluate equation (29). However, a more conservative estimate is obtained by using the lower-bound values of $P\{Z = u\}$ when evaluating the approximation to κ_{13} since then the right side is (relatively) maximized. Figure 6 shows comparisons of the estimated actual values of σ_X with the approximations found using equation (29) and also with

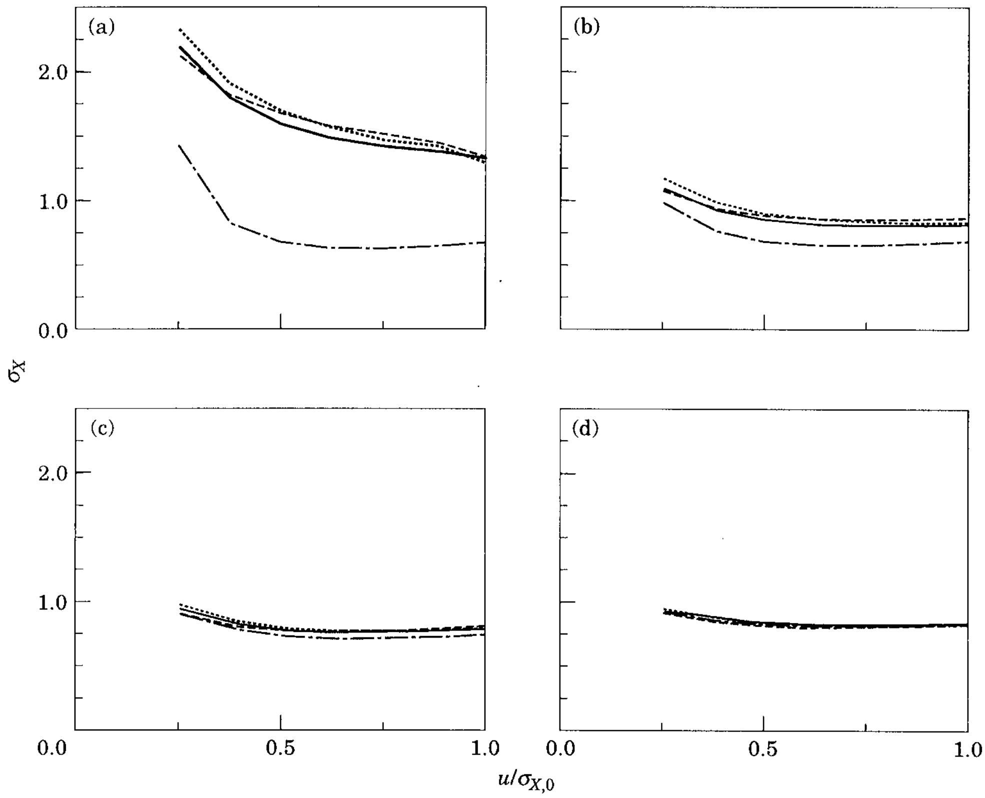

Figure 6. Comparison of estimated exact values of σ_X with approximations found using equation (30) and the method of stochastic averaging; Monte Carlo simulations (—); approximation 1 (· · · · ·); approximation 2 (- - -); (a) $\alpha = 0{\cdot}05$; (b) $\alpha = 0{\cdot}25$; (c) $\alpha = 0{\cdot}50$; (d) $\alpha = 0{\cdot}75$.

approximations found using the method of stochastic averaging [31]. (In the figure captions "Approximation 1" corresponds to using the value of $\sigma_{\dot{X}}$ found in "Approximation 1" above along with the minimum approximation to κ_{13}. Similarly, "Approximation 2" corresponds to using the value of $\sigma_{\dot{X}}$ found in "Approximation 2" above along with the minimum approximation to κ_{13}.) As with the approximations to $\sigma_{\dot{X}}$, the approximations to σ_X very closely agree with the estimated exact values for each value of α throughout the range of $u/\sigma_{X,0}$ values. In contrast, the results found from stochastic averaging only agree well with the estimated exact results in the cases of $\alpha = 0{\cdot}50$ and $0{\cdot}75$. In the case of $\alpha = 0{\cdot}50$, the results found from both of the approximations proposed here more closely agree with the estimated exact results in comparison with the results from stochastic averaging. In the case of $\alpha = 0{\cdot}75$, both of the approximations and the stochastic averaging method produce roughly similar results which agree very well with the estimated exact results (as is evident from Figure 6(d)). The inaccuracy of the stochastic averaging method for values of $\alpha < 0{\cdot}50$ stems from the fact that in these cases the underlying assumption of the method, namely, that the system is weakly non-linear, is violated. The system is weakly non-linear if α is close to 1 or the ratio $\sigma_{X,0}/u$ is $\ll 1$. The latter condition is not met here because low yield levels with respect to $\sigma_{X,0}$ are being considered.

8. CONCLUSIONS

Algebraic equations were derived for a bilinear hysteretic oscillator excited by stationary Gaussian white noise that related stationary response moments, conditional response moments, and the probability of being in the plastic state. Then, considering the physics of the oscillator at low yield level upcrossings, *a priori* bounds, verified by Monte Carlo simulations, were assigned to the conditional response moments $\mathrm{E}[\dot{X}|Z = u]$ and $\mathrm{E}[\dot{X}^2|Z = u]$. These bounds in conjunction with *a priori* approximations of $\mathrm{E}[XZ]$ and $\mathrm{E}[Z^2]$ led to an equation expressed in terms of known system constants, bounded parameters, and the probability of being in the plastic state, $P\{Z = u\}$, that was then solved numerically to determine bounds on $P\{Z = u\}$. With bounds on $P\{Z = u\}$ established, very accurate approximations of $\sigma_{\dot{X}}$ and σ_X were evaluated as functions of $P\{Z = u\}$.

The primary usefulness of the derived expressions is that they provide a quick and easy way to determine very accurate second order stationary response statistics of a bilinear hysteretic oscillator for a wide array of system parameters, including highly non-linear cases (whether due to low yields levels, small values of α, or both). These stationary statistics, in turn, may be used in rudimentary design or risk analysis considerations. Further, given the values σ_X, $\sigma_{\dot{X}}$ and $\sigma_Z = \sqrt{\kappa_{33}}$, it may be possible to approximate higher order moments and joint probability density functions using a Gram–Charlier series with a Minai–Suzuki modification [32, 33, 18].

The advantage of using the expressions proposed herein is that they provide very accurate approximations even for cases in which other approximate methods, such as the method of stochastic averaging, fail due to severe non-linearity. Moreover, an added benefit of using the derived expressions is that in order to determine the stationary statistics of the displacement and velocity, bounds on the probability of being in the plastic state have to be computed. These bounds on the (stationary) probability of being in the plastic state potentially may be useful quantities for purposes of assessing reliability of the system as the reliability of the system most surely depends on the system's excursions into the plastic state.

ACKNOWLEDGMENT

The first author is grateful for the support provided through the research program *Dynamics of Structures* at Aalborg University, Aalborg, Denmark.

REFERENCES

1. J. B. Roberts 1978 The response of an oscillator with bilinear hysteresis to stationary random excitation *ASME Journal of Applied Mechanics* **45**, 923–928.
2. J. B. Roberts 1980 The yielding behavior of a randomly excited elasto–plastic structure. *Journal of Sound and Vibration* **72**, 71–85.
3. J. B. Roberts 1988 In *Nonlinear Stochastic Dynamic Engineering Systems* (editors F. Ziegler and G. I. Schueller). Berlin, Heidelberg: Springer-Verlag; 361–379. Application of averaging methods to randomly excited hysteretic systems.

4. J. B. ROBERTS 1981 *Shock and Vibration Digest* **13**, 17–28. Response of nonlinear mechanical systems to random excitation; part I: Markov methods.
5. W. D. IWAN and P.-T. SPANOS 1978 *ASME Journal of Applied Mechanics* **45**, 170–174. Response envelope statistics for nonlinear oscillators with random excitation.
6. P.-T. SPANOS 1979 *Journal of the Acoustical Society of America* **65**, 404–410. Hysteretic structural vibrations under random load.
7. B. F. SPENCER JR. 1986 *Reliability of Randomly Excited Hysteretic Structures*. Berlin, Heidelberg: Springer-Verlag.
8. T. K. CAUGHEY 1960 *ASME Journal of Applied Mechanics* **27**, 640–643. Random excitation of a system with bilinear hysteresis.
9. T. K. CAUGHEY 1971 *Advances in Applied Mechanics* **11**, 209–253. Nonlinear theory of random vibrations.
10. T. K. CAUGHEY 1986 *Probabilistic Engineering Mechanics* **1**, 2–4. On the response of non-linear oscillators to stochastic excitation.
11. W. D. IWAN and L. D. LUTES 1968 *Journal of the Acoustical Society of America* **43**, 545–552. The response of a bilinear hysteretic system to stationary random excitation .
12. L. D. LUTES 1970 *Journal of the Acoustical Society of America* **48**, 299–306. Approximate technique for treating random vibration of hysteretic systems.
13. W. D. IWAN and J. YANG 1972 *ASME Journal of Applied Mechanics* **39**, 545–550. Application of statistical linearization techniques to nonlinear multidegree-of-freedom systems.
14. T. T. BABER and Y. K. WEN 1981 *ASCE Journal of Engineering Mechanics* **107**, 1069–1087. Random vibration of hysteretic degrading systems.
15. Y. K. WEN 1980 *ASME Journal of Applied Mechanics* **47**, 150–154. Equivalent linearization for hysteretic systems under random excitation.
16. R. L. GROSSMAYER and W. D. IWAN 1981 *Earthquake Engineering and Structural Dynamics* **9**, 171–185. A linearization scheme for hysteretic systems subjected to random excitation.
17. J. B. ROBERTS 1981 *Shock and Vibration Digest* **13**, 15–29.
18. S. R. K. NIELSEN, K. J. MØRK and P. THOFT-CHRISTENSEN 1990 *Structural Safety* **8**, 369–379. Reliability of hysteretic systems subjected to white noise excitation.
19. D. KARNOPP and T. D. SCHARTON 1966 *Journal of the Acoustical Society of America* **39**, 1154–1161. Plastic deformation in random vibration.
20. E. H. VANMARCKE and D. VENEZIANO 1974 *Proceedings of the 5th World Conference on Earthquake Engineering*, *Rome*, 2851–2863. Probabilistic seismic response of simple inelastic systems.
21. N. R. IYENGAR and J. K. IYENGAR 1978 *ASCE Journal of Engineering Mechanics Division* **104**, 383–397. Stochastic analysis of yielding system.
22. R. GROSSMAYER 1979 *Journal of Sound and Vibration* **65**, 353–379. Elastic–plastic oscillators under random excitation.
23. R. GROSSMAYER 1981 *ASCE Journal of Engineering Mechanics* **39**, 97–115. Stochastic analysis of elasto–plastic systems.
24. F. ZIEGLER and H. IRSCHIK 1985 *Proceedings of the 8th International SMIRT Conference* **M1**, 123–128. Nonstationary random vibrations of yielding beams.
25. H. IRSCHIK 1986 *Acta Mechanica* **60**, 265–280. Nonstationary random vibration of yielding multi-degree-of-freedom systems: methods of effective envelope functions.
26. B. K. BHARTIA and E. H. VANMARCKE 1991 *ASCE Journal of Engineering Mechanics* **117**, 2407–2428. Associate linear system approach to nonlinear random vibration.
27. O. DITLEVSEN 1986 *ASCE Journal of Engineering Mechanics* **112**, 386–406. Elasto–plastic oscillator with Guassian excitation.
28. O. DITLEVSEN 1991 *Journal of Sound and Vibration* **145**, 443–456. Gaussian excited elasto–plastic oscillator with rare visits to the plastic domain.

29. O. Ditlevsen and L. Bognar 1993 *Probabilistic Engineering Mechanics* **8**, 209–231. Plastic displacement distributions of the Gaussian white noise excited elasto–plastic oscillator.
30. M. K. Kaul and J. Penzien 1974 *ASCE Journal of the Engineering Mechanics Division* **100**, 1025–1038. Stochastic seismic analysis of yielding offshore towers.
31. J. B. Roberts and P. D. Spanos 1990 *Random Vibrations and Statistical Linearization*. Chichester: John Wiley.
32. R. Minai and Y. Suzuki 1985 *Proceedings ROC-Japan Joint Seminar on Multiple Hazards Mitigation, National Taiwan University, ROC*, 193–208. Seismic reliability analysis of building structures.
33. Y. Suzuki and R. Minai 1985 *Proceedings of the 4th International Conference on Structural Safety and Reliability* **2**, 177–186. Seismic reliability analysis of hysteretic structures based on stochastic differential equations.